Medical Imaging

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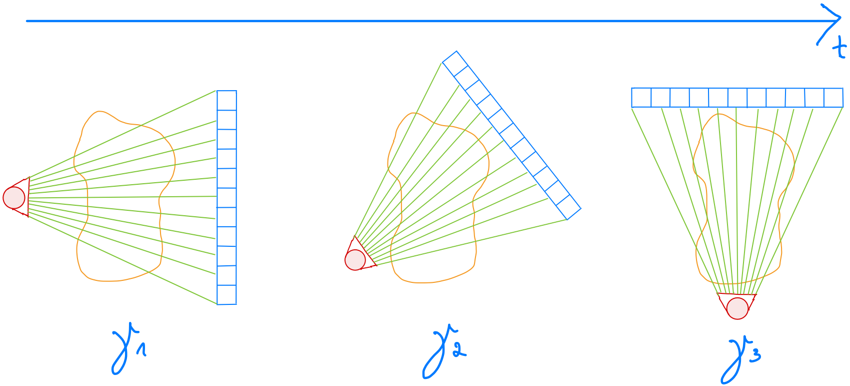
Computed Tomography

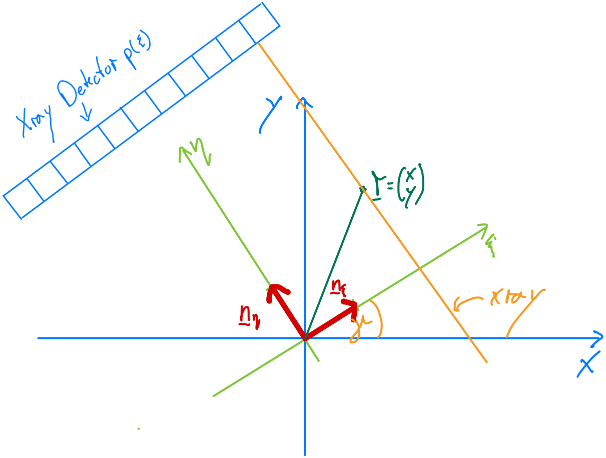
Computed Tomography

The aim of Computed Tomography is to reconstruct $\mu(x, y)$ from given detector data $p(\xi)$.

Basic idea of CT: Rotate the X-ray source and the detector (the so called gantry) around the object.



We differentiate the patient coordinate system (x, y) and the coordinate system of the gantry (ξ, η)



The unit vectors of the (ξ, η) coordinate system are given by

$$\boldsymbol{n}_\xi = \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix}$$

$$\boldsymbol{n}_\eta = \begin{pmatrix} -\sin\gamma \\ \cos\gamma \end{pmatrix}$$

Change of basis

One can convert $(x, y) =: \boldsymbol{r}$ coordinates into (ξ, η) coordinates using orthogonal projections

$$\xi = \langle \boldsymbol{r}, \boldsymbol{n}_\xi \rangle = (x, y)^T \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix} = x\cos\gamma + y\sin\gamma$$

$$\eta = \langle \boldsymbol{r}, \boldsymbol{n}_\eta \rangle = (x, y)^T \begin{pmatrix} -\sin\gamma \\ \cos\gamma \end{pmatrix} = -x\sin\gamma + y\cos\gamma$$

In matrix-vector form

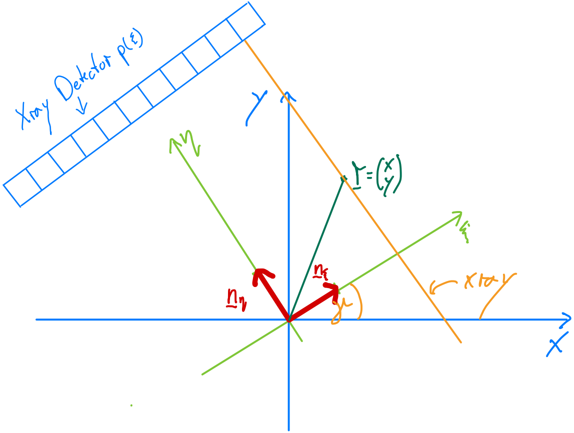
$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{R}_\gamma \begin{pmatrix} x \\ y \end{pmatrix}$$

Since $\boldsymbol{R}_\gamma$ is orthogonal we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \boldsymbol{R}_\gamma^\mathsf{T} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi\cos\gamma - \eta\sin\gamma \\ \xi\sin\gamma + \eta\cos\gamma \end{pmatrix}$$

Imaging Sequence

During a CT measurement the gantry is rotated by 180° or 360° around the patient and the projection data $p(\xi, \gamma)$ are measured. The goal of CT is to reconstruct $\mu(x, y)$ given the projection data $p(\xi, \gamma)$.



Radon Transform

The detector data $p(\xi, \gamma)$ can be calculated via the integration of $\mu(x, y)$ along an X-ray. The X-ray is described by a path $\boldsymbol{\delta}_{\xi,\gamma} : [a, b] \to \mathbb{R}^2$ at position ξ with angle γ and interval boundaries $a \in \mathbb{R}$ at the source and $a < b \in \mathbb{R}$ at the detector. The relation is mathematically described by the Radon transform R:

$$\begin{aligned} p(\xi, \gamma) &= R\{\mu(x, y)\} \\ &= \int_{\boldsymbol{\delta}_{\xi,\gamma}} \mu(x, y)\,\mathrm{d}s \end{aligned}$$

The parametrization of the X-ray is given by

$$\boldsymbol{\delta}_{\xi,\gamma}(\eta) = \boldsymbol{R}_\gamma^\mathsf{T} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \cos\gamma - \eta \sin\gamma \\ \xi \sin\gamma + \eta \cos\gamma \end{pmatrix}.$$

Radon Transform

Thus, we can calculate the line integral with

$$\begin{aligned} p(\xi, \gamma) &= R\{\mu(x, y)\} \\ &= \int_{\boldsymbol{\delta}_{\xi,\gamma}} \mu(x, y) \, \mathrm{d}s \\ &= \int_a^b \mu(\boldsymbol{\delta}_{\xi,\gamma}(\eta)) \left\| \boldsymbol{\delta}'_{\xi,\gamma}(\eta) \right\|_2 \, \mathrm{d}\eta \\ &= \int_a^b \mu(\xi \cos\gamma - \eta \sin\gamma, \xi \sin\gamma + \eta \cos\gamma) \, \mathrm{d}\eta \end{aligned}$$

since

$$\|\boldsymbol{\delta}'_{\xi,\gamma}(\eta)\|_2 = \left\| \begin{pmatrix} \sin\gamma \\ \cos\gamma \end{pmatrix} \right\|_2 = 1.$$

Radon Transform

Without loss of generality we can set $a = -\infty$ and $b = \infty$ since $\mu(x, y)$ can be assumed to be zero outside the circle covered by the CT system. Thus, the Radon transform reads

$$p(\xi, \gamma) = \int_{-\infty}^{\infty} \mu(\xi \cos\gamma - \eta \sin\gamma, \xi \sin\gamma + \eta \cos\gamma) \, \mathrm{d}\eta$$

Remark
The Radon transform and the question about its invertability have been investigated by Johann Radon already in 1917, without any concrete application in mind.

Johann Radon: Über die Bestimmung von Funktionen längs gewisser Mannigfaltigkeiten. In: Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-Physische Klasse. Band 69, 1917, S. 262–277.

Radon Transform as an Inverse Problem

Compare the inverse problems of radiography and CT:

Radiography

$$p(\xi) = \int_{-\infty}^{\infty} \mu(\xi, \eta) \, \mathrm{d}\eta$$

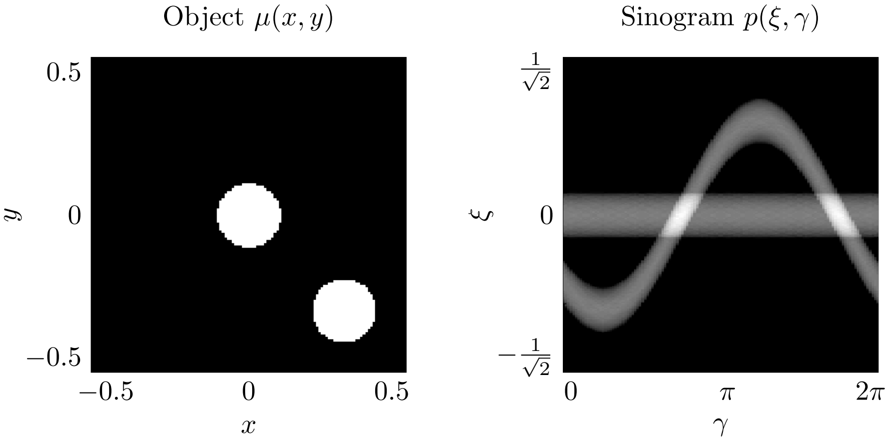
Computed Tomography

$$p(\xi, \gamma) = \int_{-\infty}^{\infty} \mu(\xi \cos\gamma - \eta \sin\gamma, \xi \sin\gamma + \eta \cos\gamma) \, \mathrm{d}\eta$$

Key Observation: The radiography imaging operator maps from a 2D into a 1D space and thus looses information. The CT operator maps from a 2D space into a 2D space and thus might preserve all information.

Sinogram

The raw data $p(\xi, \gamma)$ is also named a sinogram and can be displayed as an image. It is called a sinogram due to the sinus shaped structures.



Fourier Slice Theorem

The Fourier Slice theorem provides answers the fundamental question if it is possible to reconstruct $\mu(x, y)$ from given $p(\xi, \gamma)$, i.e. it shows that the Radon operator is bijective/invertible.

Theorem
Let $P(q, \gamma) := \mathcal{F}_{1D}\{p(\xi, \gamma)\}$ *and* $F(u, v) := \mathcal{F}_{2D}\{\mu(x, y)\}$. *Furthermore let*

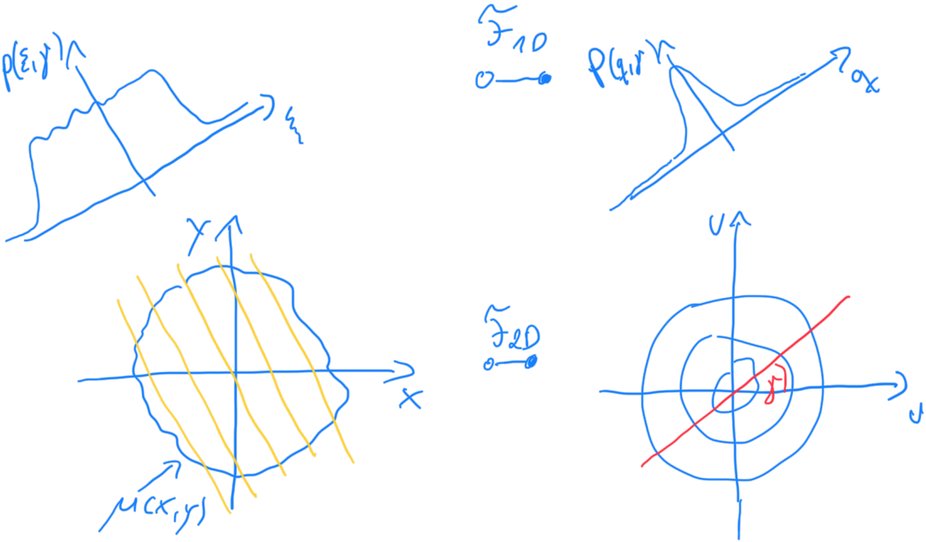
$$u = q \cos \gamma$$

$$v = q \sin \gamma$$

Then it holds that

$$F(u, v) = F(q \cos \gamma, q \sin \gamma) = P(q, \gamma)$$

Fourier Slice Theorem



Proof

$$
\begin{aligned}
P(q,\gamma) &= \int_{-\infty}^{\infty} p(\xi,\gamma)\mathrm{e}^{-2\pi\mathrm{i}\xi q}\,\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(\xi\cos\gamma-\eta\sin\gamma,\xi\sin\gamma+\eta\cos\gamma)\mathrm{e}^{-2\pi\mathrm{i}\xi q}\,\mathrm{d}\eta\,\mathrm{d}\xi
\end{aligned}
$$

We now make the coordinate transform

$$
\begin{aligned}
x &= \xi\cos\gamma-\eta\sin\gamma \\
y &= \xi\sin\gamma+\eta\cos\gamma
\end{aligned}
$$

Using the Jacobian determinant we have

$$
\begin{aligned}
\mathrm{d}x\,\mathrm{d}y &= \left|\det\frac{\partial(x,y)}{\partial(\xi,\eta)}\right|\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \left|\det\begin{pmatrix}\cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma\end{pmatrix}\right|\,\mathrm{d}\xi\,\mathrm{d}\eta = 1\,\mathrm{d}\xi\,\mathrm{d}\eta
\end{aligned}
$$

Proof

Thus we have

$$\begin{aligned} P(q,\gamma) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)\mathrm{e}^{-2\pi\mathrm{i}(x\cos\gamma + y\sin\gamma)q}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)\mathrm{e}^{-2\pi\mathrm{i}(x(q\cos\gamma) + y(q\sin\gamma))}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y)\mathrm{e}^{-2\pi\mathrm{i}(ux+vy)}\,\mathrm{d}x\,\mathrm{d}y \\ &= F(q\cos\gamma, q\sin\gamma) = F(u,v), \end{aligned}$$

which completes the proof.

Remarks

The Fourier slice theorem answers the question if $\mu(x, y)$ can be reconstructed from $p(\xi, \gamma)$ in the continuous case.

Answer
It can be reconstructed for any $\mu(x, y)$ for which the continuous Fourier transform $F(u, v) := \mathcal{F}_{2D}\{\mu(x, y)\}$ exists. A sufficient criterion for this is that μ is a function of the Hilbert space $L_2(\mathbb{R}^2)$, i.e. μ has to fulfill

$$\|\mu(x, y)\|_2 := \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\mu(x, y)|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}} < \infty$$

Analytic Image Reconstruction

- The Fourier slice theorem allows to *analytically* solve the inverse problem of determining the tomographic image by direct inversion of the imaging operator.
- This in turn yields a *direct* image reconstruction method.
- Methods that instead tackle the inverse problem in its original form are often named *algebraic image reconstruction* (ART) methods.

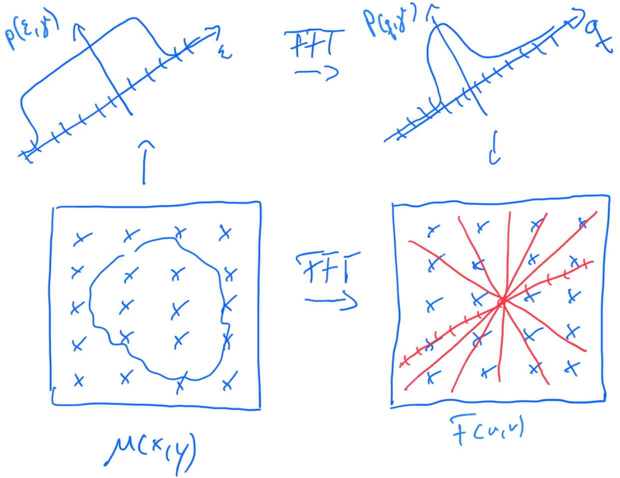
Fourier Based Reconstruction

Using the Fourier slice theorem one can derive the following direct reconstruction algorithm

1. $\forall \gamma$ calculate $P(q, \gamma) = \mathcal{F}_{1D}\{p(\xi, \gamma)\}$
2. $\forall u = q \cos \gamma, v = q \sin \gamma$ calculate $F(u, v) = P(q, \gamma)$
3. calculate $\mu(x, y) := \mathcal{F}_{2D}^{-1}\{F(u, v)\}$

Practical Issue

In the discrete setting the Fourier transforms are realized using the FFT. However, since the FFP is only applicable for **equidistant** node points one has the situation that the points (u, v) and $(q \cos \gamma, v = q \sin \gamma)$ do not match.



Practical Issue

Consequently, the Fourier based reconstruction has to resample the data in Fourier space using e.g. interpolation techniques. Interpolation in Fourier space leads, however, to larger numerical errors in image space.

$\Rightarrow$ Fourier based reconstruction usually not used in todays CT scanners.

Remark
Nowadays fast FFT for **non-equidistant** node points are known. Will be discussed later in the lecture.

Filtered Backprojection

- Standard reconstruction technique used in CT scanners
- Uses Fourier slice theorem (as well)

Derivation

Expressing $\mu(x, y)$ in terms of its inverse Fourier transform $F(u, v)$:

$$\mu(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) \mathrm{e}^{2\pi \mathrm{i}(ux + vy)} \, \mathrm{d}u \, \mathrm{d}v$$

We now perform a coordinate transform from Cartesian into polar coordinates:

$$u = q \cos \gamma$$
$$v = q \sin \gamma$$

Derivation

The Jacobian determinant is given by

$$\begin{aligned}
\mathrm{d}u\,\mathrm{d}v &= \left|\det \frac{\partial(u,v)}{\partial(q,\gamma)}\right| \,\mathrm{d}q\,\mathrm{d}\gamma \\
&= \left|\det \begin{pmatrix} \frac{\partial u}{\partial q} & \frac{\partial v}{\partial q} \\ \frac{\partial u}{\partial \gamma} & \frac{\partial v}{\partial \gamma} \end{pmatrix}\right| \,\mathrm{d}q\,\mathrm{d}\gamma \\
&= \left|\det \begin{pmatrix} \cos\gamma & \sin\gamma \\ -q\sin\gamma & q\cos\gamma \end{pmatrix}\right| \,\mathrm{d}q\,\mathrm{d}\gamma \\
&= |q\cos^2\gamma + q\sin^2\gamma|\,\mathrm{d}q\,\mathrm{d}\gamma \\
&= |q|\,\mathrm{d}q\,\mathrm{d}\gamma
\end{aligned}$$

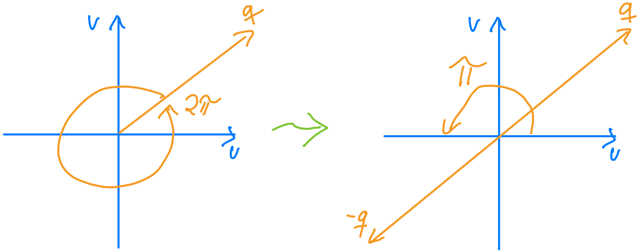
Derivation

Thus, we have

$$\mu(x, y) = \int_0^{2\pi}\int_0^{\infty} F(q\cos\gamma, q\sin\gamma)e^{2\pi i(xq\cos\gamma + yq\sin\gamma)}|q|\,dq\,d\gamma$$

$$\underset{\text{FS theorem}}{=} \int_0^{2\pi}\int_0^{\infty} P(q, \gamma)e^{2\pi i(xq\cos\gamma + yq\sin\gamma)}|q|\,dq\,d\gamma$$

Changing the integral limits



Derivation

yields

$$\mu(x, y) = \int_0^{\pi} \int_{-\infty}^{\infty} P(q, \gamma) \mathrm{e}^{2\pi \mathrm{i} q (x \cos\gamma + y \sin\gamma)} |q| \, \mathrm{d}q \, \mathrm{d}\gamma$$

The inner integral can be defined to be a function $h(\xi, \gamma)$:

$$h(\xi, \gamma) = \int_{-\infty}^{\infty} P(q, \gamma) |q| \mathrm{e}^{2\pi \mathrm{i} q \xi} \, \mathrm{d}q$$

$$\mu(x, y) = \int_0^{\pi} h(x \cos\gamma + y \sin\gamma, \gamma) \, \mathrm{d}\gamma$$

Algorithm

With this we now can formulate the filtered backprojection algorithm (input $p(\xi, \gamma)$, output $\mu(x, y)$):

1. $\forall\gamma$ calculate $P(q, \gamma) = \mathcal{F}_{1D}\{p(\xi, \gamma)\} = \int_{-\infty}^{\infty} p(\xi, \gamma)\mathrm{e}^{-2\pi \mathrm{i} q\xi}\,\mathrm{d}\xi$
2. $\forall\gamma$ calculate $h(\xi, \gamma) = \int_{-\infty}^{\infty} P(q, \gamma)|q|\mathrm{e}^{2\pi \mathrm{i} q\xi}\,\mathrm{d}q$
3. calculate $\mu(x, y) = \int_0^{\pi} h(x\cos\gamma + y\sin\gamma, \gamma)\,\mathrm{d}\gamma$

Remarks

- The first two steps of the algorithm apply the filter $|q|$ in Fourier space. $|q|$ is a high pass or edge filter.
- Instead of applying the filter in Fourier space one can alternatively apply it directly in spatial domain.

Issue
What is the Fourier transform of $|q|$?

Fourier transform of $|q|$

Consider:

$$w_\varepsilon(\xi) = \frac{\varepsilon^2 - (2\pi\xi)^2}{(\varepsilon^2 - (2\pi\xi)^2)^2} \quad \circ\!\!-\!\!\bullet \quad |\boldsymbol{q}|\mathrm{e}^{-\varepsilon|\boldsymbol{q}|}$$

In the limit $\varepsilon \to 0$ one obtains

$$|\boldsymbol{q}|\mathrm{e}^{-\varepsilon|\boldsymbol{q}|} \to |\boldsymbol{q}|$$

and

$$w_0(\xi) = -\frac{1}{(2\pi\xi)^2}$$

Fourier transform of $|q|$

$\Rightarrow$ If the Fourier integrals converge (depends on $p(\xi, \gamma)$!) we can apply the filter in image space via a convolution:

$$h(\xi, \gamma) = (p(\tilde{\xi}, \gamma) * w_0(\tilde{\xi}))(\xi)$$

Since $w_0(\xi)$ has "local" support (after truncation), the convolution can be effectively applied in image space. This has been done in first generation CTs.

Backprojection

Lets have a look at the inner part of the filtered backprojection, i.e. the integration

$$f(x, y) = \int_0^{\pi} h(x \cos \gamma + y \sin \gamma, \gamma) \, \mathrm{d}\gamma$$

Here,

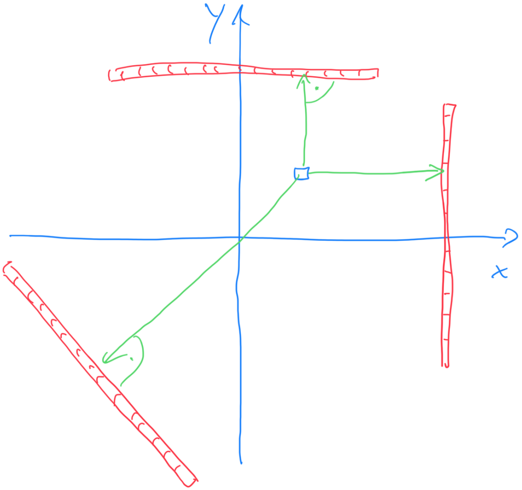
$$\xi = x \cos \gamma + y \sin \gamma$$

describes a line within $\mathbb{R}^2$.

There are now two interpretation of the integration

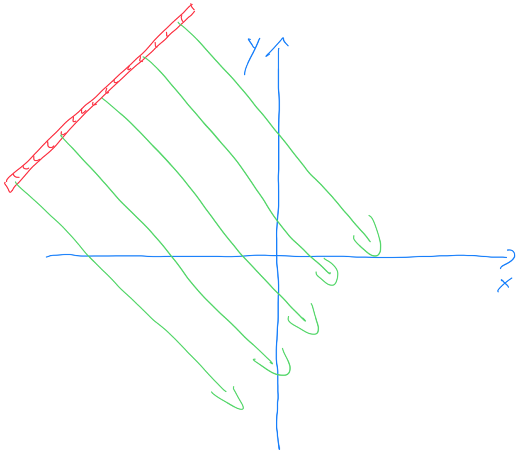
Backprojection – Interpretation 1

- One selects a certain pixels x, y
- Then, for each angle γ one throws the shortest line to the detector and pics the value.
- Thus, pixels are successively filled.



Backprojection – Interpretation 2

- Entire projection is projected back in a single step over the entire xy plane.
- The backprojected data are added to an image buffer.



Interpolation

Remark In a discrete setting the filtered backprojection requires an interpolation step.

$$\underbrace{h}_{\text{sampled at equidistant nodes}} \left(\underbrace{x \cos\gamma + y \cos\gamma}_{\text{In dependence of } \gamma \text{ non-regular}}, \gamma \right)$$

This interpolation is uncritical, since it happens in spatial domain so that numerical errors have only local effects.

Discretization

Number of angles

$L \in \mathbb{N}$: $\gamma_l = \frac{l}{L}\pi, \quad l = 0, \ldots, L-1$

Number of detector pixels

$M \in \mathbb{N}$: $\xi_m = A\left(\frac{m+0.5}{M} - 0.5\right), \quad m = 0, \ldots, M-1$

where A is the size of the detector.

Number of image pixels

$N_x \in \mathbb{N}$: $x_{n_x} = \Omega_x\left(\frac{n_x+0.5}{N_x} - 0.5\right), \quad n_x = 0, \ldots, N_x-1$

$N_y \in \mathbb{N}$: $y_{n_y} = \Omega_y\left(\frac{n_y+0.5}{N_y} - 0.5\right), \quad n_y = 0, \ldots, N_y-1$

where Ω_x and Ω_y are the side lengths of the image.

Time Complexity

Filtered Backprojection

$$\mathcal{O}(LM \log M + N^2 L) \underset{\text{if } L \approx N \approx M}{=} \mathcal{O}(N^2 \log N + N^3) = \mathcal{O}(N^3)$$

Fourier Slice based Reconstruction

$$\mathcal{O}(N^2 \log N)$$

Thus, FBP is a little bit slower, which is usually not critical in practice.

Parallel Implementation

The FBP can be impemented in a massively parallelized fashion. In particular reconstruction can already start *during* data acquisition.

$\rightarrow$ low latency

Filtering

In practice the projection $p(\xi, \gamma)$ are affected by noise

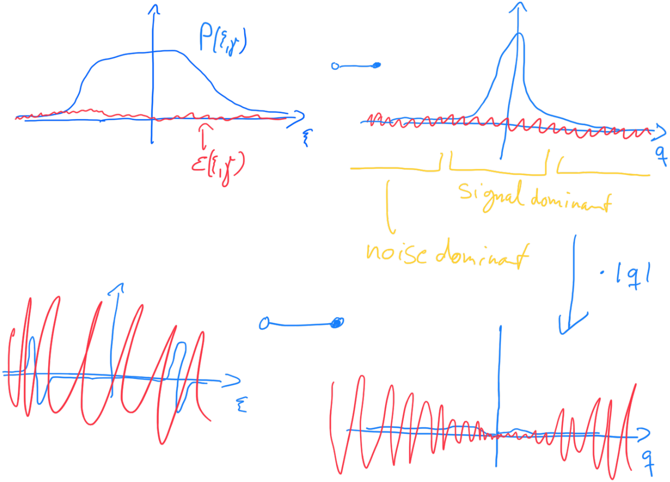
$$p(\xi, \gamma) = p^{\text{true}}(\xi, \gamma) + \varepsilon(\xi, \gamma)$$

The ramp filter $|p|$ leads to a noise amplification since the noise $\mathcal{F}\{\varepsilon(\xi, \gamma)\}$ is frequency independent.

$\rightarrow$ It is thus important to band-limit the filter

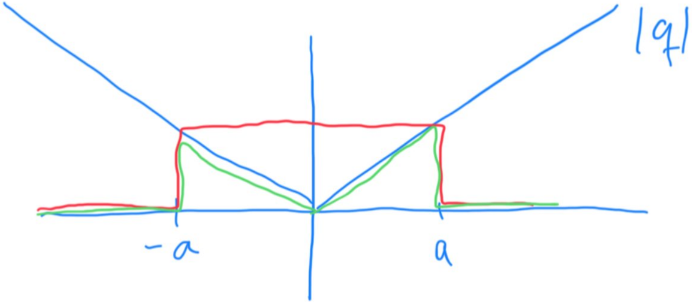
Filtering

- Illustration of the noise amplification of analytical image reconstruction.
- While low frequency components are signal dominated, high frequencies are noise dominated (upper right).
- High pass filter amplifies noise (lower left and right).



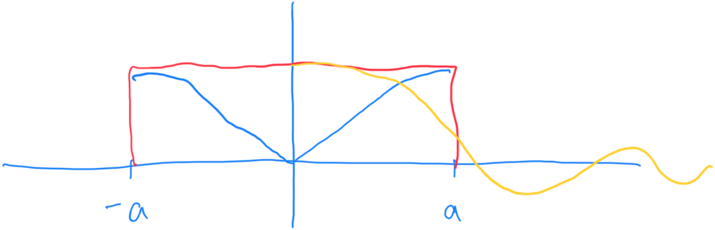
Ram-Lak Filter (Ramachandran - Lakshminarayan)

Replace $|q|$ with $|q|\text{rect}(\frac{q}{2a})$



Shepp-Logan Filter

Replace $|q|$ with $|q|\mathrm{rect}(\frac{q}{2a})\mathrm{sinc}(\frac{q}{a})$



Summary

- CT is an extension of radiography where the gantry is rotated.
- This solves the *uniqueness* issue of radiography and in turn allows for solving the inverse problem.
- The imaging operator can be *analytically* inverted and allows for direct image reconstruction.
- The inversion of the Radon transform is *noise amplifying*. This can mitigated by *filtering*.