Medical Imaging

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Inverse Problems

Inverse Problems

In most tomographic imaging methods the task of reconstructing a slice/volume image of the object is an *inverse problem*.

Let I be a multi-dimensional function describing the unknown image, O be a function that describes the *raw measurement data* collected with a tomographic device and S be an operator that maps I to O. Then, the imaging equation for any tomographic imaging method can be written in the form

$$O = S(I). \tag{1}$$

Before we dive into tomography, we discuss the key terminology of inverse problems.

Direct Problem

- Given: The input / cause (i) for a system (S)
- Task: Determine the output of the system

$$o = S(i) \tag{2}$$

Examples:

- Given a current in a electromagnetic coil with a defined geometry. Calculate the magnetic field in space that is generated by the current.
- Given some object within the bore of a tomographic device. Calculate the signals, the device will measure.

Inverse Problem

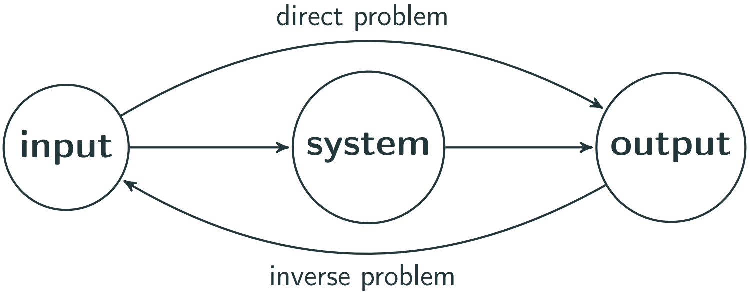
- Given: The output of a system o (i.e. usually some noisy measurements)
- Task: Determine the input to the system i such that

$$S(i) \approx o \tag{3}$$

Examples:

- Given the magnetic field at a finite number of spatial positions. Determine the coil geometry / current that could have been the cause for the observations.
- Given some measurements from a tomographic device. Calculate the object within the scanner bore.

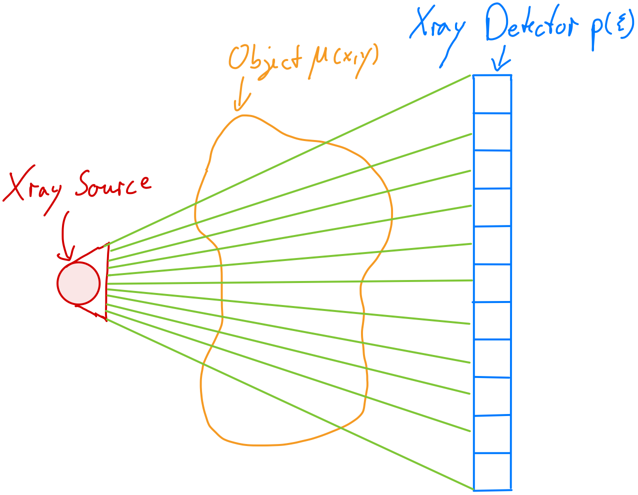
Overview



Radiography

Radiography

During a *radiography* the object under examination is illuminated with X-ray.



When the ray passes the object it will be damped/attenuation due to interactions with the matter of the object. In particular the ray is absorbed and scattered. The attenuation coefficient μ is given by

$$\mu = \mu_{\mathsf{S}} + \mu_{\mathsf{A}}$$

where μ_{S} is the scattering coefficient and μ_{A} is the absorption coefficient. The unit of μ is $\frac{1}{m}$. μ is spatially dependent and thus we consider it to be a function $\mu : \mathbb{R}^3 \to \mathbb{R}_+$

Attenuation in Homogeneous Medium

Let $I : \mathbb{R} \to \mathbb{R}_+$ be the intensity of the X-ray. Let it pass along the η axis. Then one observes

$$\begin{aligned} I(\eta + \Delta\eta) &= I(\eta) - \mu\Delta\eta I(\eta) \\ \Leftrightarrow I(\eta + \Delta\eta) - I(\eta) &= -\mu\Delta\eta I(\eta) \\ \Leftrightarrow \frac{I(\eta + \Delta\eta) - I(\eta)}{\Delta\eta} &= -\mu I(\eta) \end{aligned}$$

When considering the limit $\Delta\eta \to 0$ one obtains

$$\lim_{\Delta\eta \to 0} \frac{I(\eta + \Delta\eta) - I(\eta)}{\Delta\eta} = \frac{\mathrm{d}I}{\mathrm{d}\eta} = -\mu I(\eta),$$

which is an ordinary differential equation.

Attenuation in Homogeneous Medium

By separation of variables one obtains

$$\frac{\mathrm{d}I}{I} = -\mu \,\mathrm{d}\eta$$

Integration yields

$$\int \frac{1}{I}\,\mathrm{d}I = \int -\mu \,\mathrm{d}\eta$$

and in turn

$$\ln |I| = -\mu\eta + c.$$

Exponentiation leads to

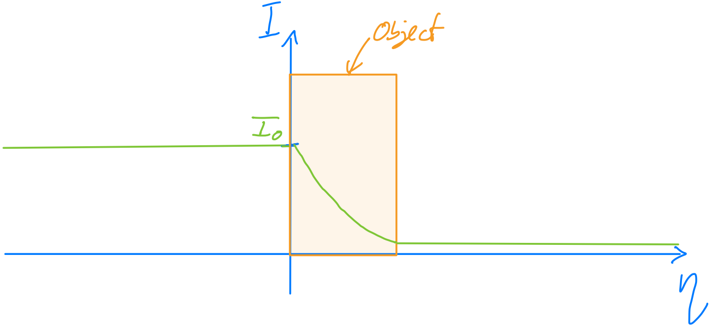
$$I(\eta) = \tilde{c}\mathrm{e}^{-\mu\eta}$$

Attenuation in Homogeneous Medium

Using the initial condition $I(0) = I_0$ which is the X-ray intensity at the source one obtains the **Lambert-Beer law**

$$I(\eta) = I_0 \mathrm{e}^{-\mu\eta}$$

Note that the Lambert-Beer law is only fulfilled for homogeneous media where μ is constant.



Attenuation in Inhomogeneous Medium

In an inhomogeneous medium μ depends on η so that

$$\frac{\mathrm{d}I}{I} = -\mu(\eta)\,\mathrm{d}\eta.$$

Integration leads to

$$\int \frac{1}{I}\,\mathrm{d}I = -\int \mu(\eta)\,\mathrm{d}\eta$$

so that

$$\ln|I| = -\int \mu(\eta)\,\mathrm{d}\eta + c.$$

Attenuation in Inhomogeneous Medium

Exponentiation leads to

$$I(\eta) = \tilde{c} \exp\left(-\int \mu(\eta)\,\mathrm{d}\eta\right).$$

Using the initial condition $I(0) = I_0$ one obtains

$$I(\eta) = I_0 \exp\left(-\int \mu(\eta)\,\mathrm{d}\eta\right).$$

Attenuation in Inhomogeneous Medium

We now only consider the intensity at the detector

$$I_{\mathrm{D}} = I(\eta_{\mathrm{D}}) = I_0 \exp\left(-\int_0^{\eta_{\mathrm{D}}} \mu(\eta)\,\mathrm{d}\eta\right)$$

Dividing by I_0 and taking the logarithm leads to

$$\ln(I_{\mathrm{D}}/I_0) = -\int_0^{\eta_{\mathrm{D}}} \mu(\eta)\,\mathrm{d}\eta =: -p$$

Here p is the so-called *projection*.

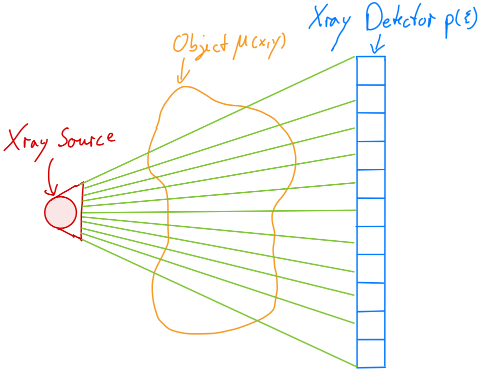
Remarks

- The intensity I_D measured at the detector has always to be related to the intensity I_0 and the X-ray source.
- Typically X-ray data is visualized in the logarithmic form $p = -\ln(I_D/I_0)$.
- In X-ray and CT systems that source intensity can be usually adjusted to generate different contrasts.

Geometries

Until now we have considered a single X-ray passing through the medium and being detected with a single detector pixel.

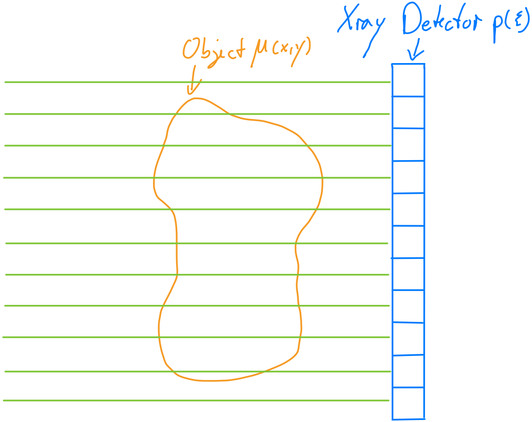
In practice the source emits the X-ray in a the form of a fan (2D) or a cone (3D).



The detected projection value is in the case of a fan beam X-ray source a 1D function $p : \mathbb{R} \to \mathbb{R}$.
In classical radiography cone beam is used and the detector is a 2D function $p : \mathbb{R}^2 \to \mathbb{R}$.

Parallel Beam Geometry

A major simplification that we make from now on is that the X-ray source is moved to $-\infty$ yielding the so-called *parallel beam geometry*.



Radiography as an Inverse Problem

We next consider radiography as an inverse problem. The system equation for the 2D setting in parallel beam geometry reads

$$p(\xi) = \int_0^{\eta_D} \mu(\eta, \xi) \, \mathrm{d}\eta.$$

where $p : \mathbb{R} \to \mathbb{R}_+$ are the measured projections and $\mu : \mathbb{R}^2 \to \mathbb{R}_+$ is the attenuation coefficient.

Direct Problem

The direct problem is easily solvable. After discretization one just has to sum up the values of μ along the beam line.

Inverse Problem

The inverse problem reads: Given p, determine μ. Is that problem solvable?

Existence of a Solution

A solution does exist. For instance a trivial solution is

$$\mu_{\text{trivial}}(\eta, \xi) := \frac{p(\xi)}{\eta_D}$$

since

$$\int_0^{\eta_D} \mu_{\text{trivial}}(\eta, \xi)\,\mathrm{d}\eta = \int_0^{\eta_D} \frac{p(\xi)}{\eta_D}\,\mathrm{d}\eta = \left[\eta \frac{p(\xi)}{\eta_D}\right]_0^{\eta_D} = \eta_D \frac{p(\xi)}{\eta_D} = p(\xi).$$

Uniqueness of a Solution

The existence of a solution is a necessary condition but is that solution unique? Lets consider

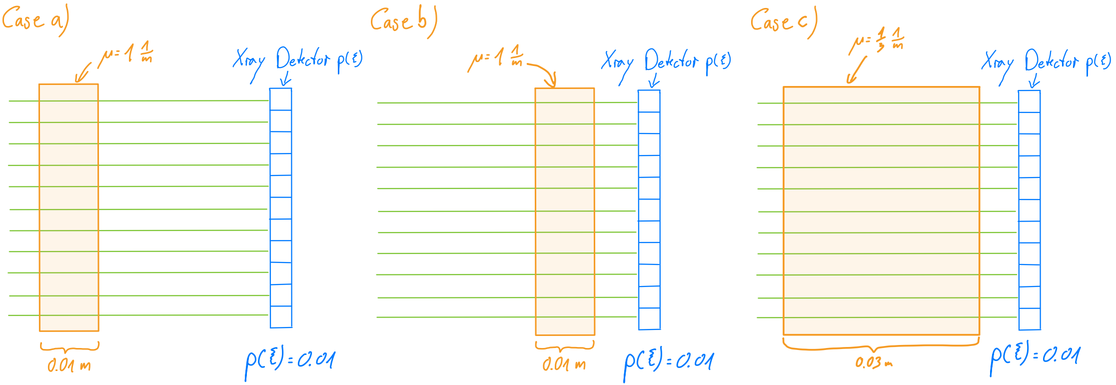
$$\mu_\beta(\eta,\xi) := \begin{cases} \frac{p(\xi)}{\beta} & \eta \leq \beta \\ 0 & \eta > \beta \end{cases}$$

with $\beta \in (0, \eta_D]$, which yields

$$\int_0^{\eta_D} \mu_\beta(\eta,\xi)\,\mathrm{d}\eta = \int_0^{\beta} \frac{p(\xi)}{\beta}\,\mathrm{d}\eta = \left[\eta\frac{p(\xi)}{\beta}\right]_0^{\beta} = \beta\frac{p(\xi)}{\beta} = p(\xi).$$

Thus, the inverse problem has infinite solutions. In practice this means that this particular inverse problem is not solvable, i.e. it is not possible to determine $\mu(\xi,\eta)$ from $p(\xi)$.

Limitations of X-ray imaging



Summary

- Radiography allows to project the attenuation coefficients along a certain direction.
- During this process *depth information* is lost.
- The inverse problem of determining the attenuation coefficients μ from the projections is not solvable. Therefore, in practice, the medical doctor looks at the projection images and tries to *decompose* it by incorporating *prior knowledge* of the underlying anatomy.